

The First Thirty Years of Large- N Gauge Theory

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Abstract

I review some developments in the large- N gauge theory since 1974. The main attention is paid to: multicolor QCD, matrix models, loop equations, reduced models, 2D quantum gravity, free random variables, noncommutative theories, AdS/CFT correspondence.

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Preface

Large- N gauge theories are with us since the work by 't Hooft of 1974 [1].¹ In this talk I review some milestones of their developments since then. I pay the main attention to multicolor QCD, matrix models, loop equations, reduced models, 2D quantum gravity, free random variables, noncommutative theories, AdS/CFT correspondence. These issues attracted the most interest of the community over the last thirty years.

I apology that some of the important results, in particular on QCD phenomenology and supersymmetric gauge theories, are not mentioned in this short talk. Correspondingly, the list of references is far from being complete. An extended description of the subject of this talk as well as more references can be found in my recent book [2].

1974: Multicolor QCD

The effective coupling constant in quantum chromodynamics (QCD) becomes large at large distances where the perturbation theory is not applicable. The idea of 't Hooft [1] was to consider the dimensionality of the gauge group $SU(N_c)$ as a parameter and to perform an expansion in $1/N_c$, the inverse number of colors. The motivation was an expansion in the inverse number of field components N in statistical mechanics, where only bubble graphs of the type depicted in Fig. 1 survive at large N .

The expansion of QCD in $1/N_c$ (known as the $1/N_c$ -expansion) rearranges diagrams of perturbation theory according to their topology. Only planar diagrams of the type depicted in Fig. 2 survive the large- N_c limit, while the expansion in $1/N_c$ plays the role of a topological expansion. In the 't Hooft limit when $\lambda = g^2 N_c$ is kept fixed as $N_c \rightarrow \infty$, a generic (properly normalized) diagram of genus h with L quark loops and B external boundaries behaves as

$$\text{generic graph} \sim \left(\frac{1}{N_c}\right)^{2h+L+2(B-1)} \quad (1)$$

independently of the order of the diagram in the coupling constant.

This is similar to an expansion in the string coupling constant in dual-resonance models of the strong interaction, that also has a topological character and the phenomenological consequences of which agree with experiment. The accuracy of the leading-order term, which is often called multicolor QCD or large- N_c QCD, is expected to be of the order of the ratios of meson widths to their masses, *i.e.* about 10–15%.

The simplification of QCD in the large- N_c limit arises from the fact that the number of planar graphs grows with the number of vertices only exponentially rather than factorially as do the total number of graphs. The number of graphs of genus h with n_0 vertices grows at large n_0 as

$$\#_h(n_0) \approx e^{\Lambda_c n_0} (n_0)^{-b_h}, \quad (2)$$

¹Here and below I quote the year of a journal publication, as it was custom in the pre-arXiv epoch.

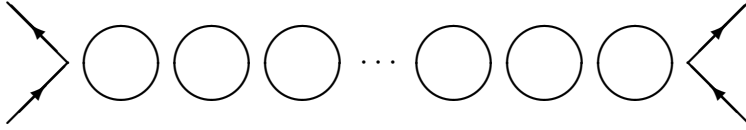


Figure 1: Bubble graph which survives the large- N limit of N -component vector models in statistical mechanics.

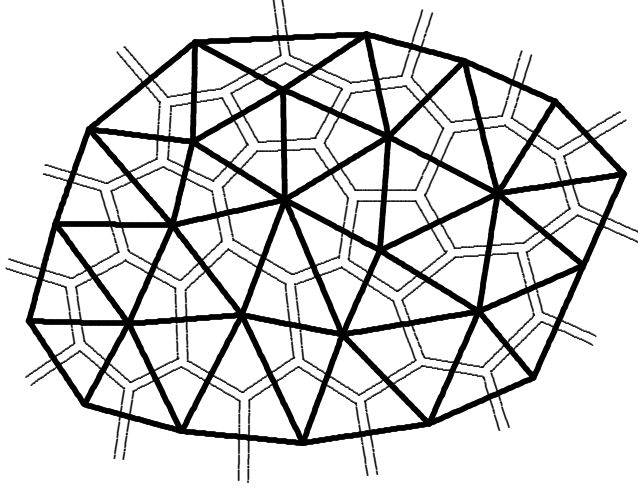


Figure 2: Planar diagram (depicted by double lines) which survives the large- N_c limit of $N_c \times N_c$ matrix models with cubic interaction. Its dual graph (depicted by bold lines) is constructed from equilateral triangles.

where Λ_c is a constant, so the dependence on genus resides only in the index b_h in the pre-exponential.

While QCD is simplified in the large- N_c limit, it is still not yet solved (except in $d = 2$ dimensions [3]). Generically, it is a problem of infinite matrices, rather than of infinite vectors as in the theory of second-order phase transitions in statistical mechanics. Since the correlators of gauge-invariant operators factorize in the large- N_c limit, it looks like the leading-order term of a “semiclassical” WKB-expansion in $1/N_c$. This fact was linked to the possible existence of a master field [4] describing the “classical” $N_c = \infty$ limit.

1978: One-matrix model

Matrix models first appeared in statistical mechanics and nuclear physics and turned out to be very useful in the analysis of various physical systems where the energy levels of a complicated Hamiltonian can be approximated by the distribution of eigenvalues of a random matrix. The statistical averaging is then replaced by averaging over an appropriate ensemble of random matrices.

Matrix models possess some features of multicolor QCD but are simpler and can often

be solved as $N_c \rightarrow \infty$ (*i.e.* in the planar limit) using the methods proposed for multicolor QCD. For the simplest case of the Hermitian one-matrix model, which is related to the problem of enumeration of graphs, an explicit solution at large N_c was first obtained by Brézin, Itzykson, Parisi and Zuber [5]. It inspired a lot of activity on this subject, in particular the methods to construct the genus expansion in $1/N_c$ were developed.

The Hermitian one-matrix model is defined by the partition function

$$Z_{1h} = \int d\varphi e^{-N_c \text{tr} V(\varphi)}, \quad (3)$$

where $d\varphi$ is the measure for integrating over Hermitian $N_c \times N_c$ matrices. A very important property of the model is that $\text{tr} V(\varphi)$ depends only on the eigenvalues of the matrix φ . Similarly, representing φ in a canonical form $\varphi = U P U^\dagger$ with unitary $N_c \times N_c$ matrix U and diagonal $P = \text{diag} \{p_1, \dots, p_{N_c}\}$, the measure $d\varphi$ can be written in a standard Weyl form

$$d\varphi = dV \prod_{i=1}^{N_c} dp_i \Delta^2(P), \quad (4)$$

where $\Delta(P) = \prod_{i < j} (p_i - p_j)$ is the Vandermonde determinant. The contribution from angular degrees of freedom residing in U factorizes, so the partition function (3) is expressed via N_c degrees of freedom. The integral can therefore be calculated as $N_c \rightarrow \infty$ using the saddle-point method.

The saddle-point equation simplifies for the (normalized) spectral density $\rho(p)$ which describes the distribution of eigenvalues of the matrix φ and becomes a continuous non-negative function of p as $N_c \rightarrow \infty$. Then the integral is dominated by a saddle-point configuration which obeys the equation [5]

$$V'(p) = 2 \oint d\lambda \frac{\rho(\lambda)}{p - \lambda} \quad \boxed{p \in \text{support of } \rho}, \quad (5)$$

where the RHS involves the principal part of the integral. Equation (5) holds only when p belongs to the support of ρ .

For a general potential $V(p)$, the simplest solution is when $\rho(p)$ has support on a single interval $[a, b]$. This looks similar to Wigner's semicircle law for the Gaussian case which is perturbed by the interactions. Such a spectral density equals

$$\rho(p) = \frac{M(p)}{2\pi} \sqrt{(p - a)(b - p)}, \quad (6)$$

where a and b are the ends of the support and $M(p)$ is a polynomial of degree $K-2$ if $V(p)$ is a polynomial of degree K . This solution was first obtained in [5] for cubic and quartic potentials.

The one-cut solution (6) is acceptable if $M(p)$ is not negative in the interval $[a, b]$ which always happens for small values of the interaction couplings g_3, g_4 , *etc.* With increasing couplings, a third-order phase transition may occur after which a more complicated multicut solution is realized.

1979: Loop equations

The loop-space approach in QCD was motivated by Wilson's lattice formulation of non-Abelian gauge theories. It is based on the fact that all observables can be expressed at large N_c via quantum averages of the trace of a non-Abelian phase factor (= the Wilson loop)

$$W(C) = \left\langle \frac{1}{N_c} \text{tr} \mathbf{P} e^{ig \oint_C dx^\mu A_\mu(x)} \right\rangle. \quad (7)$$

Remarkably, in the large- N_c limit this $W(C)$ satisfies a closed equation derived in 1979 by Migdal and me [6] and known as the loop equation.

The simplest form of the loop equation I know uses the functional Laplacian Δ which is defined as the $\epsilon \rightarrow 0$ limit of the second-order variational operator

$$\Delta^{(\epsilon)} = \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 e^{-|\int_{\sigma_1}^{\sigma_2} d\sigma \sqrt{\dot{x}^2(\sigma)}|/\epsilon} \frac{\delta}{\delta x_\mu(\sigma_1)} \frac{\delta}{\delta x_\mu(\sigma_2)}. \quad (8)$$

The loop equation then reads as

$$\Delta W(C) = \lambda \oint_C dx_\mu \int_x^x dy_\mu \delta^{(d)}(x-y) W(C_{xy}) W(C_{yx}), \quad (9)$$

where C_{xy} and C_{yx} are the parts of the contour C from x to y and y to x , respectively.

The operator $\Delta^{(\epsilon)}$ defined by Eq. (8) can be inverted so that the loop equation (9) with the proper choice of boundary conditions can be transformed to the form

$$W[x] = 1 - \frac{1}{2} \int_0^\infty dA \left\{ \left\langle J[x + \sqrt{A}\xi] \right\rangle_\xi^{(\epsilon)} - \left\langle J[\sqrt{A}\xi] \right\rangle_\xi^{(\epsilon)} \right\}, \quad (10)$$

where $J[x]$ stands for the functional on the RHS of Eq. (9). In Eq. (10) the average over the loops $\xi(\sigma)$ is given by the path integral

$$\left\langle F[\xi] \right\rangle_\xi^{(\epsilon)} = \frac{\int_{\xi(0)=\xi(1)} D\xi e^{-S} F[\xi]}{\int_{\xi(0)=\xi(1)} D\xi e^{-S}} \quad (11)$$

with the reparametrization-invariant local action

$$S = \frac{1}{4} \int_0^1 d\sigma \left\{ \frac{\epsilon}{\sqrt{\dot{x}^2(\sigma)}} \dot{\xi}^2(\sigma) + \frac{\sqrt{\dot{x}^2(\sigma)}}{\epsilon} \xi^2(\sigma) \right\} \quad (12)$$

which is of the type of that for a harmonic oscillator.

The integral form (10) of the loop equation is most convenient for an iterative solution in λ , which reproduces the diagrams of perturbation theory as $\epsilon \rightarrow 0$. The diagram with a three-gluon vertex appears thereby as a result of doing an uncertainty of the type $\epsilon \times 1/\epsilon$.

As far as nonperturbative solutions of the loop equation are concerned, it was shown that the area law is a self-consistent solution for asymptotically large smooth contours. Although this is consistent with the string representation, it was also shown that the free

bosonic Nambu–Goto string is *not* a solution for intermediate loops. A formal solution for all loops was found [7] in the form of an elf-string with two-dimensional elementary fermions living in the surface. They were introduced to provide a factorization. For large loops the internal fermionic structure becomes frozen, so the area law is recovered. However, it is unclear whether or not the elf-string is practically useful for studies of multicolor QCD, since the methods of dealing with the string theory in four dimensions have not yet been developed.

1982: Reduced models

The large- N_c reduction was discovered in 1982 by Eguchi and Kawai [8] and stated that the $SU(N_c)$ gauge theory on a d -dimensional space-time is equivalent at $N_c = \infty$ to the one at a single point. The continuum reduced action²

$$S_{\text{EK}}[\mathbf{A}] = - \left(\frac{2\pi}{\Lambda} \right)^d \frac{1}{4g^2} \text{tr} [\mathbf{A}_\mu, \mathbf{A}_\nu]^2, \quad (13)$$

can be viewed as obtained from the usual one by substituting the covariant derivative according to

$$i\partial_\mu + gA_\mu(x) \xrightarrow{\text{red.}} D^\dagger(x) \mathbf{A}_\mu D(x), \quad (14)$$

where the matrix \mathbf{A}_μ is space-independent. The Eguchi–Kawai (EK) model was found as a solution of the (lattice) loop equation.

The equivalence of the EK model and the usual theory was based on an extra R^d symmetry of the reduced action (13), which should not be broken spontaneously. Soon after the EK model was proposed, it was recognized that for $d > 2$ a phase transition occurs in the EK model with decreasing coupling constant and the R^d symmetry is, in fact, broken at weak couplings. The quenching prescription [9], when the eigenvalues of the (infinite) Hermitian matrix \mathbf{A}_μ are quenched, was proposed to cure the construction. The quenched EK model results in a reduced model which recovers multicolor QCD both on the lattice and in the continuum, while it makes sense only for planar diagrams.

An elegant alternative to the quenched EK model, which also preserves the R^d symmetry and describes multicolor QCD, was proposed in 1983 by González-Arroyo and Okawa [10] on the basis of a twisting reduction prescription. The corresponding lattice version of the twisted Eguchi–Kawai model (TEK) lives on a unit hypercube with twisted boundary conditions. The continuum version of TEK [11] is described by the action

$$S_{\text{TEK}}[\mathbf{A}] = - \left(\frac{2\pi}{\Lambda} \right)^d \frac{1}{4g^2} \text{tr} ([\mathbf{A}_\mu, \mathbf{A}_\nu] + iB_{\mu\nu}\mathbf{1})^2, \quad (15)$$

where $B_{\mu\nu}$ is an antisymmetric constant tensor. It cannot be omitted in Eq. (15) because \mathbf{A}_μ are infinite matrices (= Hermitian operators) for which $\text{tr} [\mathbf{A}_\mu, \mathbf{A}_\nu] \neq 0$. Although

²Here the coefficient $(2\pi/\Lambda)^d$ represents a “unit volume” for a box.

the actions (13) and (15) look similar, the difference between EK and TEK resides in the vacuum states which are determined by the equation

$$[\mathbf{A}_\mu, \mathbf{A}_\nu] = -iB_{\mu\nu}\mathbf{1} \quad (16)$$

and drastically differ from those at $B_{\mu\nu} = 0$ given by diagonal (commutative) \mathbf{A}_μ .

The TEK model reveals interesting mathematical structures associated with representations of the Heisenberg commutation relation (16) (in the continuum) or its finite-dimensional approximation by unitary matrices (on the lattice). In contrast to the quenched EK models which describe only planar graphs, the TEK models make sense order by order in $1/N_c$ and even at finite N_c , when they are associated with gauge theories on a noncommutative lattice described below.

While the reduced models look like a great simplification, since the space-time is reduced to a point, they still involve an integration over d infinite matrices which is a continual path integral. For some years it was not clear whether or not this is a real simplification of the original theory that can make it solvable, so the point of view on the reduced models was that they are just an elegant representation at large N_c .

1985 & 1990: 2D quantum gravity

Matrix models are generically associated [12] with discretization of random surfaces. The simplest Hermitian one-matrix model corresponds to a zero-dimensional embedding space, *i.e.* to two-dimensional Euclidean quantum gravity described by the partition function

$$Z_{2\text{DG}} = \int \mathcal{D}g \, e^{-\int d^2x \sqrt{g} \Lambda + 2(1-h)/\mathcal{G}}. \quad (17)$$

Here Λ denotes the cosmological constant, while the coupling \mathcal{G} weights topologies of the 2D world. The path integral in Eq. (17) is over all metrics $g_{\mu\nu}(x)$.

The idea of dynamical triangulation of random surfaces is to approximate a surface by a set of equilateral triangles. The coordination number (the number of triangles meeting at a vertex) does not necessarily equal six, which represents internal curvature of the surface. The partition function (17) is then approximated by

$$Z_{\text{DT}} = \sum_h e^{2(1-h)/\mathcal{G}} \sum_{T_h} e^{-\Lambda n_t}, \quad (18)$$

where the sum over triangles is split into the sum over genus h and the sum over all possible triangulations T_h at fixed h . In (18) n_t denotes the number of triangles which is not fixed at the outset, but rather is a dynamical variable.

The partition function (18) can be represented as a matrix model. A graph dual to a generic set of equilateral triangles coincides with a graph in the Hermitian one-matrix model with a cubic interaction as is depicted in Fig. 2. The precise statement is that Z_{DT} equals the (logarithm of the) partition function (3) with $N_c = \exp(1/\mathcal{G})$ and the cubic

coupling constant $g_3 = \exp(-\Lambda)$. This can be easily shown by comparing the graphs. The logarithm is needed to pick up connected graphs in the matrix model. Analogously, the interaction $\text{tr } \varphi^k$ in the matrix model is associated with discretization of random surfaces by regular k -gons, the area of which is $k-2$ times the area of the equilateral triangle.

Continuum limits of the Hermitian one-matrix model are reached at the points of phase transitions. While no phase transition is possible at finite N_c since the system has a finite number of degrees of freedom, it may occur as $N_c \rightarrow \infty$ which plays the role of a statistical limit. This third-order phase transition is of the type discovered by Gross and Witten [13] for lattice QCD in $d = 2$ which reduces to a unitary one-matrix model. It is associated with divergence of the sum over graphs at each fixed genus rather than with divergence of the sum over genera. The contribution of a graph with n_0 trivalent vertices is $\sim (-g_3)^{n_0}$ but an entropy (= the number) of such graphs at fixed genus is given by Eq. (2), so the sum can diverge at a certain critical value $g_3 = \exp(-\Lambda_c)$. This critical behavior emerges when one or more roots of $M(p)$ in the spectral density (6) approaches the end points a or b .

This continuum limit is good for planar diagrams or genus zero, while higher genera are still suppressed as N_c^{-2h} . A description of the higher genera by the matrix models became possible after the ‘‘October breakthrough’’ of 1990 [14]. It was based on the fact that the effective parameter of the genus expansion near the critical point is

$$\mathcal{G} = \frac{1}{N_c^2 (\Lambda - \Lambda_c)^{5/2}} \quad (19)$$

and can be made finite if $(\Lambda - \Lambda_c) \sim N_c^{-4/5}$ as $N_c \rightarrow \infty$. This special limit, when the couplings reach critical values in a N_c -dependent way simultaneously with $N_c \rightarrow \infty$, is called the double scaling limit.

The double scaling limit of the Hermite an one-matrix model gave the genus expansion of 2D quantum gravity [14]. An extension of this construction to multimatrix models made it possible to describe 2D quantum gravity interacting with matter. However, nobody managed so far to extend these results to higher dimensions (beyond the so-called $d = 1$ barrier).

1995: Free random variables

Since the paper [1] there were several attempts to formulate what I would call the planar quantum field theory, *i.e.* to describe the planar limit in a field-theoretical language. Because external legs of a planar graph cannot be interchanged (except in a cyclic order), this leads us to the concept of noncommutative variables.

The planar quantum field theory possesses a number of unusual properties. In particular, the usual exponential relation between the generating functionals W and Z for connected graphs and all graphs does not hold for the planar graphs. The reason for this is that exponentiation of a connected planar diagram can give disconnected nonplanar diagrams. The required relation for planar graphs can be constructed [15] by means of

introducing noncommutative sources $\mathbf{j}_\mu(x)$. “Noncommutative” means that there is no way to transform $\mathbf{j}_\mu(x) \mathbf{j}_\nu(y)$ into $\mathbf{j}_\nu(y) \mathbf{j}_\mu(x)$.

The conjugate variable obeys

$$\frac{\delta}{\delta \mathbf{j}_\mu(x)} \mathbf{j}_\nu(y) = \delta_{\mu\nu} \delta^{(d)}(x - y). \quad (20)$$

There would be a commutator in this formula for Bosons or an anticommutator for Fermions, but it is just like that in the planar limit! It is called the Cuntz algebra and it results in neither Bose nor Fermi but Boltzmann statistics.

The planar contribution to the Green functions and their connected counterparts can be obtained, respectively, from the generating functionals $Z[\mathbf{j}]$ and $W[\mathbf{j}]$ by applying the noncommutative derivative according to Eq. (20) which picks up only the leftmost variable. The usual exponential relation is superseded by

$$Z[\mathbf{j}] = W[\mathbf{j} Z[\mathbf{j}]], \quad (21)$$

while the cyclic symmetry gives $W[\mathbf{j} Z[\mathbf{j}]] = W[Z[\mathbf{j}] \mathbf{j}]$. In other words, given $W[\mathbf{j}]$, one should construct a function $\mathbf{J}_\mu[\mathbf{j}]$ which is inverse to

$$\mathbf{j}_\mu(x) = \mathbf{J}_\mu(x) \frac{1}{W[\mathbf{J}]}, \quad (22)$$

after which Eq. (21) gives $Z[\mathbf{j}] = W[\mathbf{J}]$.

For the one-matrix case when there is only one source (that commutes with itself), Wigner’s semicircle law as well as some results of Ref. [5] are reproduced by this technique. For the d -dimensional Gaussian case when $W[\mathbf{j}]$ is quadratic in \mathbf{j} , $Z[\mathbf{j}]$ can be expressed via a continued fraction. However, in general, mathematical methods for dealing with functions of this kind of noncommutative variables are not developed.

Nevertheless, there is an example of two-dimensional Yang–Mills theory, whose solution at $N_c = \infty$ was described by Singer [16] using an adequate mathematical language of free random variables introduced in noncommutative probability theory by Voiculescu. Such a master field quantifies earlier ideas about the one-matrix model [17] and a stochastic master field [18]. In some cases, an explicit solution of a d -dimensional planar quantum field theory can be easily obtained from its $d = 0$ or $d = 1$ counterparts, using either additivity or multiplicativity of the free random variables [16].

1998: Noncommutative theories

The recent revival of the reduced models has arisen from the M(atr)ix formulation [19] of M-theory combining all types of superstring theories. The novel point of view on the reduced models, discovered in 1998 by Connes, Douglas and Schwarz [20], was their equivalence to gauge theories on noncommutative space, whose coordinates do not commute and obey the commutation relation

$$[\mathbf{x}_\mu, \mathbf{x}_\nu] = i \theta_{\mu\nu} \mathbf{1}. \quad (23)$$

The multiplication of matrices (operators) can be represented in the coordinate space, introducing a noncommutative product of functions

$$\mathbf{f} \cdot \mathbf{g} \Rightarrow f(x) * g(x) \stackrel{\text{def}}{=} f(x) \exp \left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta_{\mu\nu} \partial_\nu \right) g(x), \quad (24)$$

where $\overleftarrow{\partial}_\mu$ acts on $f(x)$ and ∂_ν acts on $g(x)$. This star-product is noncommutative but associative similarly to the product of matrices (operators).

The action (15) can then be rewritten as the action of the noncommutative $U_\theta(1)$ gauge theory:³

$$S[\mathcal{A}] = \frac{1}{4\lambda} \int d^d x \mathcal{F}^2, \quad (25)$$

where $\lambda = g^2 N_c$ coincides with the 't Hooft coupling of the TEK model. The gauge field $\mathcal{A}_\mu(x)$, $\mathbf{A}_\mu \Rightarrow (i\partial_\mu + \mathcal{A}_\mu)$, is no longer matrix-valued but rather noncommutativity of matrices in the reduced model is transformed into noncommutativity of coordinates in the noncommutative gauge theory. In Eq. (25) \mathcal{F} denotes the noncommutative field strength

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i(\mathcal{A}_\mu * \mathcal{A}_\nu - \mathcal{A}_\nu * \mathcal{A}_\mu). \quad (26)$$

Note that cubic and quartic interactions of \mathcal{A}_μ enter the action (25) quite similarly to Yang–Mills theory!

The diagrams of the perturbation-theory expansion of the noncommutative theories look similar to those in Yang–Mills theory. Planar diagrams do not depend on the parameter of noncommutativity θ at all, in analogy with the TEK models [10], and are the same as planar diagrams in ordinary Yang–Mills theory. For $d > 2$ the contribution of a nonplanar diagram of genus h is suppressed at large θ as [21] $(\det \theta_{\mu\nu})^{-h}$, so only the planar diagrams survive as $\theta \rightarrow \infty$. For this reason it is often said that the noncommutative $U_\infty(1)$ gauge theory is a master field of multicolor QCD.

Remarkably, the analogy between the TEK models and the noncommutative theories can be pursued beyond the $N_c = \infty$ or $\theta = \infty$ limits. The genus expansion of the noncommutative $U_\theta(1)$ gauge theory can be reproduced by the TEK models in a certain double scaling limit [22]. Moreover, the TEK models at finite N_c are mapped [23] onto noncommutative theories on a finite periodic lattice (= a discrete torus). These results are an extension of the fact that noncommutative gauge theories on a torus are equivalent at special values of $\theta_{\mu\nu}$ to ordinary Yang–Mills theories on a smaller torus with twisted boundary conditions representing the non-Abelian 't Hooft flux.

1998: AdS/CFT correspondence

A great support of the long-standing belief in the string/gauge correspondence has come recently from $\mathcal{N} = 4$ supersymmetric Yang–Mills theories (SYM). As was conjectured by Maldacena in 1998 [24], the $\mathcal{N} = 4$ SYM is equivalent to a IIB superstring in the anti-de

³The tensor $\theta_{\mu\nu}$ is inverse to $B_{\mu\nu}$ entering Eq. (15): $\theta_{\mu\nu} = B_{\mu\nu}^{-1}$.

Sitter background $AdS_5 \times S^5$. One of the motivations for this AdS/CFT correspondence was the underlying conformal symmetry of both theories. The radius R of anti-de Sitter space on the superstring side is related to the 't Hooft coupling λ on the SYM side by

$$\frac{R^2}{\alpha'} = \sqrt{\lambda}, \quad (27)$$

so the strong-coupling limit of SYM is described by supergravity in anti-de Sitter space $AdS_5 \times S^5$.

Among the most interesting predictions of the AdS/CFT correspondence for the strong-coupling limit of SYM, I shall mention the calculation [25] of the anomalous dimensions of certain operators and that [26] of the Euclidean-space rectangular Wilson loop determining the interaction potential. The former is given by the spectrum of excitations in AdS space, while the latter is given by the minimal surface formed by the worldsheet of an open string whose ends lie at the loop in the boundary of $AdS_5 \times S^5$. The computation of the Wilson loop in the supergravity approximation was also performed for circular loop, which case has then been exactly calculated [27] in SYM to all orders in λ . The result provided not only a beautiful test of the AdS/CFT correspondence at large λ but also a challenging prediction for IIB superstring in the $AdS_5 \times S^5$ background.

Yet another remarkable test of the string/gauge correspondence, which goes beyond the supergravity approximation, concerns [28] a certain class of operators in SYM, whose anomalous dimensions can be exactly computed as a function of λ both in string theory and under some mild assumptions in SYM. The exact computation in string theory is possible because the anomalous dimensions of these BMN operators correspond to the spectrum of states with large angular momentum associated with rotation of an infinitely short closed string around the equator of S^5 .

Rotating similarly a long closed folded string in AdS_5 , a very interesting prediction concerning the strong-coupling limit of the anomalous dimensions of twist (= bare dimension minus Lorentz spin n) two operators has been obtained recently in Ref. [29]:

$$\Delta - n = f(\lambda) \ln n \quad (28)$$

for large n , where

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \mathcal{O}\left((\sqrt{\lambda})^0\right) \quad (29)$$

for large λ . It has then been shown [30] how this result can be reproduced via a minimal surface of an open string spanned in the boundary by the loop with a cusp.

Equations (28) and (29) were derived [29] ignoring the S^5 part of $AdS_5 \times S^5$, which is responsible for supersymmetry, and possess the features expected for the anomalous dimension in ordinary (nonsupersymmetric) Yang–Mills theory. There are arguments for this result to be valid in ordinary Yang–Mills theory as well. This would lead us to very interesting predictions for the strong-coupling limit of QCD! A challenging problem is whether it is possible to obtain Eq. (29) within Yang–Mills theory, verifying thereby the string/gauge correspondence.

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